1N-39
393804

TECHNICAL TRANSLATION
F-25

SMALL OSCILLATIONS OF THIN RESILIENT CONICAL SHELLS

By E. I. Grigolyuk

Translated from Izvestiia Akademii Nauk SSSR,
O.T.D., no. 6, 1956

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON May 1960

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL TRANSLATION F-25

SMALL OSCILLATIONS OF THIN RESILIENT CONICAL SHELLS*

By E. I. Grigolyuk**

M. Strutt [1] investigated axially-asymmetric oscillations of thin resilient conical shells which were inelastic on their middle surfaces, fastened at one end to a stiff diaphragm and free at the other, suitable for calculation of the natural oscillation values of a conical loud-speaker. He disregarded deviation of the curve in meridional direction and allowed for inertia in all directions. The conditions of inelasticity of the middle surface of a conical shell ($\epsilon_1 = \epsilon_2 = \gamma_{12} = 0$) were met by him in the following expressions for radial w, tangential v and axial u displacement of the middle surface

$$w = -\sec\alpha\cos\omega t\sum_n\left[\frac{a}{n}\cos^2\alpha + n(r-a)\right]A_n\sin n\varphi$$

$$u = -a\cos\alpha\cos\omega t\sum_n\frac{1}{n}A_n\sin n\varphi$$

$$v = \cos\omega t\sum_n(r-a)A_n\cos n\varphi$$

where a is the radius of the smaller base of the conical shell, r is the radius of the parallel ring of the shell, φ is the polar angle in the plane of the parallel ring, $2n$ is the number of peripheral half-waves in the oscillations, ω is the frequency of the natural oscillation, t is the time, A_n is a constant and α is the angle of inclination of the generator to the parallel ring of the shell.

Here there is on the inside ring ($r = a$) a lengthening of radius $\Delta r = u\cos\alpha + w\sin\alpha = 0$ and the peripheral displacement $v = 0$. Proceeding **further, the derivation was carried out** by the energy method and a formula obtained for determination of the frequency of the natural oscillations, from which Rayleigh's formula follows by limiting transition for an **inelastic infinitely long cylindrical shell [2].**

Urk and Hut [3], to verify the applicability of Strutt's formula, made

---

*Translated from Izvestiia Akademii Nauk SSSR, O.T.D., no. 6, 1956, pp. 35-44.

**Institute of Mechanics of the Academy of Sciences, USSR.

an experimental investigation of the determination of the radial frequencies of the oscillations of thin aluminum shells with a longitudinally welded seam; at a distance of one third of the generator from the outside edge on the exterior surface of the shell there was fastened a piece of iron, opposite which there was placed an electromagnet connected to a variable frequency source, setting up an oscillation of the shell with the same frequency as the source; with oscillation of the shell, powder sprinkled on its inside surface and adhering to it is dispersed along the node line after a very short time. Thus the number of lines of junction in the oscillations is established. Non-conformity of Strutt's formula with the experimental data appeared generally in the entire diapason of frequencies. The difference in frequency was from two to three times. This is understandable if it is kept in mind that the conditions of inelasticity of the middle surface do not correspond to the real picture of the oscillations of the shell.

Attempts to take the elasticity of the middle surface of a conical shell into consideration in determining the frequencies of natural oscillations were undertaken by V. E. Breslavskii [4], who used the energy method, taking the displacement expressions in the same form as for a cylindrical shell with attached rims. The method of integration of the expression for potential energy, analogous to that taken in the literature [5] led to the circumstance that instead of a conical shell, essentially, a cylindrical shell with a certain mean radius not determined in the work was studied.

In the development of this work [6,7] we investigated the solution of the problem of small natural oscillations of thin conical shells of any pitch with attached rims; in particular, the solution for a closed conical shell and new results for a cylindrical shell [8-14] are obtained.

1. The Equation of Motion. We will solve the problem by the energy method.

The potential energy of a truncated conical shell is equal to (see figure):

$$\Pi = \frac{1}{2}\int_0^{2\pi}\int_{s_1}^{s_2}\Big\{B\Big[(\varepsilon_1+\varepsilon_2)^2 - 2(1-\mu)\Big(\varepsilon_1\varepsilon_2 - \frac{1}{4}\gamma_{12}^2\Big)\Big] + D\left[(\varkappa_1+\varkappa_2)^2 - 2(1-\mu)(\varkappa_1\varkappa_2 - \varkappa_{12}^2)\right]\Big\}\, r\, d\varphi\, ds \qquad (1.1)$$

where

$$\varepsilon_1 = \dot{u}\cos\alpha, \quad \varepsilon_2 = \frac{1}{r}(u\cos\alpha - w\sin\alpha + v')$$

$$\gamma_{12} = \frac{1}{r}(-v\cos\alpha + u' + \dot{v} r\cos\alpha),$$

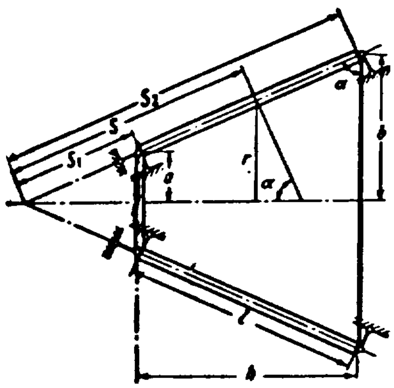
$$\varkappa_1 = \ddot{w}\cos^2\alpha, \varkappa_2 = \frac{1}{r^2}(\dot{w} r\cos^2\alpha + w'' + v'\sin\alpha) \qquad (1.2)$$

$$\varkappa_{12} = \frac{1}{r^2}(\dot{w}' r\cos\alpha + \dot{v} r\sin\alpha\cos\alpha - w'\cos\alpha - v\sin\alpha\cos\alpha)$$

$$B = \frac{E\delta}{1-\mu^2}, \qquad D = \frac{E\delta^3}{12(1-\mu^2)} \qquad (1.3)$$

whereupon $s = r / \cos\alpha$ is the distance from the apex of the shell at the

generator middle surface, $s_2 - s_1 = l$ which is the length of the shell at the generator, E and μ are the modulus of elasticity and the Poisson's ratio of the material of the shell, and δ is the thickness of the shell.



Here a dot indicates differentiation with respect to radius r and a hatchure with respect to the angle φ. The kinetic energy of the oscillating shell is

$$K = \frac{1}{2}\frac{\gamma\delta}{g}\int_0^{2\pi}\int_{s_1}^{s_2}(u_t^2 + v_t^2 + w_t^2)\, r\, d\varphi\, dr \quad (1.4)$$

where γ is the specific gravity of the material of the shell, g is the gravity acceleration. Index t below on the right signifies differentiation in time.

Let us assume that the conditions of the attached rims are fulfilled:

$$w(s_1,\varphi) = v(s_1,\varphi) = w(s_2,\varphi) = v(s_2,\varphi) = 0.$$

The following expressions for displacement fulfil these conditions:

$$\begin{aligned} w &= A_1 r^2 \sin\frac{m\pi(s-s_1)}{l}\sin n\varphi \\ v &= A_2 r^2 \sin\frac{m\pi(s-s_1)}{l}\cos n\varphi \qquad (1.5) \\ u &= A_3 r^2 \cos\frac{m\pi(s-s_1)}{l}\sin n\varphi \end{aligned}$$

Here $A_1 = A_1(t)$, $A_2 = A_2(t)$, $A_3 = A_3(t)$ are functions of time, m is the number of axial half-waves, n is the number of semi-circular waves.

The formulas (1.5) fulfil the kinematic as well as the static conditions for an oscillating cylindrical shell with attached rims.

Let us introduce expressions (1.5) into formulas (1.1) and (1.4). Then we get

$$\Pi^* = \frac{2\Pi}{\pi l^4 B} = \frac{1}{2}k_1 A_3^2 + \frac{1}{2}k_2 A_2^2 + \frac{1}{2}k_3 A_1^2 + k_4 A_2 A_3 + k_5 A_1 A_3 + k_6 A_1 A_2$$

$$K^* = \frac{2K}{\pi l^4 B} = \frac{1}{2} f_* (k_7 A_{3t}^2 + k_8 A_{2t}^2 + k_9 A_{1t}^2) \qquad \left(f_* = \frac{\gamma l^2\delta}{gB}\right) \quad (1.6)$$

where

$$k_1 = k_{10}, \qquad k_2 = k_{20} + \psi k_{21}, \qquad k_3 = k_{30} + \psi k_{31}$$

$$k_4 = k_{40}, \qquad k_5 = k_{50}, \qquad k_6 = k_{60} + \psi k_{61}, \qquad \psi = \frac{1}{12}\left(\frac{\delta}{h}\right)^2 \quad (1.7)$$

$$k_{10} = \frac{1}{24}\Big\{4\left[\frac{9}{\pi^2 m^2} + 12(1+\mu) + \pi^2 m^2\right]\cos^5\alpha +$$
$$+ 24\left[\frac{4}{\pi^2 m^2} + 8(1+\mu) + \pi^2 m^2\right]\lambda\cos^4\alpha\sin\alpha + 12\,[24(1+\mu) +$$
$$+ 5\pi^2 m^2]\,\lambda^2\cos^3\alpha\sin^2\alpha + 16\,[12(1+\mu) + 5\pi^2 m^2]\,\lambda^3\cos^2\alpha\sin^3\alpha +$$
$$+ 60\pi^2 m^2\lambda^4\cos\alpha\sin^4\alpha + 24\pi^2 m^2\lambda^5\sin^5\alpha + 3(1-\mu)\left[\left(1 + \frac{3}{\pi^2 m^2}\right)\cos^3\alpha +\right.$$
$$\left.+ 2\left(2 + \frac{3}{\pi^2 m^2}\right)\lambda\cos^2\alpha\sin\alpha + 6\lambda^2\cos\alpha\sin^2\alpha + 4\lambda^3\sin^3\alpha\right]n^2\Big\}$$

$$k_{20} = \frac{1}{12}\Big\{(1-\mu)\left[\left(3 + \pi^2 m^2 - \frac{9}{\pi^2 m^2}\right)\cos^5\alpha +\right.$$
$$+ 6\left(2 + \pi^2 m^2 - \frac{3}{\pi^2 m^2}\right)\lambda\sin\alpha\cos^4\alpha + 3(6 + 5\pi^2 m^2)\,\lambda^2\sin^2\alpha\cos^3\alpha +$$
$$+ 4(3 + 5\pi^2 m^2)\,\lambda^3\sin^3\alpha\cos^2\alpha +$$
$$\left.+ 15\pi^2 m^2\lambda^4\sin^4\alpha\cos\alpha + 6\pi^2 m^2\lambda^5\sin^5\alpha\right] + 3\left[\left(1 - \frac{3}{\pi^2 m^2}\right)\cos^3\alpha +\right.$$
$$\left.+ 2\left(2 - \frac{3}{\pi^2 m^2}\right)\lambda\sin\alpha\cos^2\alpha + 6\lambda^2\sin^2\alpha\cos\alpha + 4\lambda^3\sin^3\alpha\right]n^2\Big\}$$

$$k_{21} = \frac{\sin^4\alpha}{2}\{(1-\mu)\,[(1 + \pi^2 m^2)\cos^3\alpha + 2(1 + 2\pi^2 m^2)\,\lambda\sin\alpha\cos^2\alpha +$$
$$+ 6\pi^2 m^2\lambda^2\sin^2\alpha\cos\alpha + 4\pi^2 m^2\lambda^3\sin^3\alpha] + (\cos\alpha + 2\lambda\sin\alpha)\,n^2\}$$

$$k_{30} = \frac{\sin^2\alpha}{4}\Big\{\left(1 - \frac{3}{\pi^2 m^2}\right)\cos^3\alpha + 2\left(2 - \frac{3}{\pi^2 m^2}\right)\lambda\sin\alpha\cos^2\alpha +$$
$$+ 6\lambda^2\sin^2\alpha\cos\alpha + 4\lambda^3\sin^3\alpha\Big\}$$

$$k_{31} = \frac{\sin^2\alpha}{6}\{[9 + 12(3+\mu)\,\pi^2 m^2 + \pi^4 m^4]\cos^5\alpha +$$
$$+ 6\,[3 + 8(3+\mu)\,\pi^2 m^2 + \pi^4 m^4]\,\lambda\sin\alpha\cos^4\alpha + 3\pi^2 m^2\,[24(3+\mu) +$$
$$+ 5\pi^2 m^2]\,\lambda^2\sin^2\alpha\cos^3\alpha + 4\pi^2 m^2\,[12(3+\mu) + 5\pi^2 m^2]\,\lambda^3\sin^3\alpha\cos^2\alpha +$$
$$+ 15\pi^4 m^4\lambda^4\sin^4\alpha\cos\alpha + 6\pi^4 m^4\lambda^5\sin^5\alpha + 3\,[(-1 + \pi^2 m^2)\cos^3\alpha +$$
$$+ 2(-1 + 2\pi^2 m^2)\,\lambda\sin\alpha\cos^2\alpha + 6\pi^2 m^2\lambda^2\sin^2\alpha\cos\alpha +$$
$$+ 4\pi^2 m^2\lambda^3\sin^3\alpha]\,n^2 + 3(\cos\alpha + 2\lambda\sin\alpha)\,n^4\}$$

$$k_{40} = \frac{\pi m n}{40}\Big\{\left[4(1+\mu) + \frac{5(3-\mu)}{\pi^2 m^2}\left(2 - \frac{3}{\pi^2 m^2}\right)\right]\cos^4\alpha + 10\left[2(1+\mu) +\right.$$
$$\left.+ \frac{3(3-\mu)}{\pi^2 m^2}\right]\lambda\sin\alpha\cos^3\alpha + 10\left[4(1+\mu) + \frac{3(3-\mu)}{\pi^2 m^2}\right]\lambda^2\sin^2\alpha\cos^2\alpha +$$
$$+ 40(1+\mu)\,\lambda^3\sin^3\alpha\cos\alpha + 20(1+\mu)\,\lambda^4\sin^4\alpha\Big\}$$

$$k_{50} = \frac{\pi m\sin\alpha}{20}\Big\{\left(4\mu + \frac{10}{\pi^2 m^2} - \frac{15}{\pi^4 m^4}\right)\cos^4\alpha + 10\left(2\mu + \frac{3}{\pi^2 m^2}\right)\lambda\sin\alpha\cos^3\alpha +$$
$$+ 10\left(4\mu + \frac{3}{\pi^2 m^2}\right)\lambda^2\sin^2\alpha\cos^2\alpha + 40\mu\lambda^3\sin^3\alpha\cos\alpha + 20\mu\lambda^4\sin^4\alpha\Big\}$$

$$k_{60} = \frac{n\sin\alpha}{4}\Big\{\left(1 - \frac{3}{\pi^2 m^2}\right)\cos^3\alpha + 2\left(2 - \frac{3}{\pi^2 m^2}\right)\lambda\sin\alpha\cos^2\alpha +$$
$$+ 6\lambda^2\sin^2\alpha\cos\alpha + 4\lambda^3\sin^3\alpha\Big\}$$

$$k_{61} = \frac{n\sin^3\alpha}{4}\{[-\mu + (2-\mu)\,\pi^2 m^2]\cos^3\alpha + 2\,[-\mu +$$
$$+ 2(2-\mu)\,\pi^2 m^2]\,\lambda\sin\alpha\cos^2\alpha + 6(2-\mu)\,\pi^2 m^2\lambda^2\sin^2\alpha\cos\alpha +$$
$$+ 4(2-\mu)\,\pi^2 m^2\lambda^3\sin^3\alpha + 2(\cos\alpha + 2\lambda\sin\alpha)\,n^2\}$$

$$k_7 = \frac{1}{12}\left[\left(2 + \frac{15}{\pi^2 m^2} - \frac{45}{\pi^4 m^4}\right)\cos^5\alpha + 6\left(2 + \frac{10}{\pi^2 m^2} - \frac{15}{\pi^4 m^4}\right)\lambda\sin\alpha\cos^4\alpha +\right.$$
$$+ 30\left(1 + \frac{3}{\pi^2 m^2}\right)\lambda^2\sin^2\alpha\cos^3\alpha + 20\left(2 + \frac{3}{\pi^2 m^2}\right)\lambda^3\sin^3\alpha\cos^2\alpha +$$
$$\left.+ 30\lambda^4\sin^4\alpha\cos\alpha + 12\lambda^5\sin^5\alpha\right]$$

$$k_8 = k_9 = \frac{1}{12}\Big[\Big(2-\frac{15}{\pi^2 m^2}+\frac{45}{\pi^4 m^4}\Big)\cos^5\alpha + 6\Big(2-\frac{10}{\pi^2 m^2}+\frac{15}{\pi^4 m^4}\Big)\lambda\sin\alpha\cos^4\alpha +$$
$$+30\Big(1-\frac{3}{\pi^2 m^2}\Big)\lambda^2\sin^2\alpha\cos^3\alpha + 20\Big(2-\frac{3}{\pi^2 m^2}\Big)\lambda^3\sin^3\alpha\cos^2\alpha +$$
$$+30\lambda^4\sin^4\alpha\cos\alpha + 12\lambda^5\sin^5\alpha] \qquad (1.8)$$

$\lambda = \underline{a}/\underline{h}$, $\underline{a}$ is the radius of the upper base of the shell, $\underline{h}$ is its height.

The equation of motion is obtained from

$$\Big(\frac{\partial K^*}{\partial A_{it}}\Big)_t + \frac{\partial \Pi^*}{\partial A_i} = 0 \qquad (i = 1, 2, 3)$$

As a result we have a common system of three linear differential equations

$$\begin{aligned} f_* k_9 A_{1tt} + k_3 A_1 + k_6 A_2 + k_5 A_3 &= 0 \\ f_* k_8 A_{2tt} + k_2 A_2 + k_6 A_1 + k_4 A_3 &= 0 \\ f_* k_7 A_{3tt} + k_1 A_3 + k_5 A_1 + k_4 A_2 &= 0 \end{aligned} \qquad (1.9)$$

which is fulfilled in

$$A_1 = w_0\cos\omega t, \quad A_2 = v_0\cos\omega t, \quad A_3 = u_0\cos\omega t \qquad (1.10)$$

where w_o, v_o u_o are amplitudes of the components of displacement of the middle surface.

2. The Frequency of Natural Oscillation. Let us substitute (1.10) in (1.9) and require that the system of three linear algebraic equations give a non-null solution for w_o, v_o and u_o. This gives a cubic equation for the determination of ω_*^2

$$f(\omega_*^2) = a_3\omega_*^6 + a_2\omega_*^4 + a_1\omega_*^2 + a_0 = 0 \qquad (2.1)$$

whereupon

$$\begin{aligned} a_0 &= k_3(k_1k_2 - k_4^2) + 2k_4k_5k_6 - k_1k_6^2 - k_2k_5^2 \\ a_1 &= -k_9(k_1k_2 - k_4^2) + k_6^2k_7 - k_1k_3k_8 - k_2k_3k_7 + k_5^2k_8 \\ a_2 &= k_1k_8k_9 + k_2k_7k_9 + k_3k_7k_8, \qquad a_3 = -k_7k_8k_9 \end{aligned} \qquad (2.2)$$

Besides this

$$\omega_* = \omega l\sqrt{\frac{(1-\mu^2)\gamma}{gE}} \qquad (2.3)$$

If we disregard inertia of the shell in the axial and circumferential directions we get the following linearized formula for determination of the frequency

$$\omega_*^2 = -\frac{a_0}{a_{11}}, \qquad a_{11} = -k_9(k_1k_2 - k_4^2) \qquad (2.4)$$

This can be converted to $$\omega_*^2 = \frac{F_1 + \psi F_2 + \psi^2 F_3}{T_1 + \psi T_2} \quad (2.5)$$

where

$$F_1 = k_{10}(k_{20}k_{30} - k_{60}^2) + k_{40}(2k_{50}k_{60} - k_{30}k_{40}) - k_{20}k_{60}^2$$
$$F_2 = k_{10}(k_{20}k_{31} + k_{21}k_{30} - 2k_{60}k_{61}) + k_{40}(2k_{50}k_{61} - k_{31}k_{40}) - k_{21}k_{50}^2 \quad (2.6)$$
$$F_3 = k_{10}(k_{21}k_{31} - k_{61}^2), \qquad T_1 = (k_{10}k_{20} - k_{40}^2)k_9, \qquad T_2 = k_{10}k_{21}k_9$$

The calculations show that in formula (2.5) members with ψ^2 in the numerator and ψ in the denominator can be omitted. Hence

$$\omega_*^2 = \frac{F_1 + \psi F_2}{T_1} \quad (2.7)$$

The case of axially symmetric oscillations is not obtained from this solution because of a not completely fortunate selection of approximate functions; if it is assumed that u and w are proportional to cos $n\varphi$, the case of axially symmetrical oscillations would correspond to $n = 0$.

3. Numerical Treatment of the Results. The numerical calculations indicate that for determination of the frequencies of natural oscillation of a conical shell with an angle of inclination of the generator $\geq 75°$ the formulas to be further developed can be used for cylindrical shells of the same thickness, in which instead of the radius there should be interpolated the half-sum of the radii of the bases of the conical shell and instead of the length, the height of the conical shell.

Table 1 presents values of ω_* for a closed conical shell ($\lambda = 0$) for a series of proportions δ/b (b is the radius of the base) and for $\alpha = 3°$, 5°, 10°, 15°, 20°, 25°, 30°, 35°, 40°, 45°, 50°, 55°, 60°, 65°, 70°, 75°, 80°, 85°, 87°. The corresponding values of n are given in Table 2.

We assumed $m = 1$ and $\mu = 0.3$ in all the calculations.

4. A Cylindrical Shell. For a cylindrical shell $\alpha = 90°$, $\lambda = a/l$, where a = the radius of the middle surface, l the length of the shell. Let us introduce the designations

$$k = \frac{\psi}{\lambda^2} = \frac{1}{12}\left(\frac{\delta}{a}\right)^2, \qquad \omega^* = \lambda\omega_* = a\omega\sqrt{\frac{(1-\mu^2)\gamma}{gE}} \quad (4.1)$$

Then instead of expressions (2.1) and (2.5) we get the corresponding

$$f(\omega^{*2}) = a_3\omega^{*6} + a_2\omega^{*4} + a_1\omega^{*2} + a_0 = 0 \quad (4.2)$$

$$\omega^{*2} = \frac{f_1 + kf_2 + k^2 f_3}{t_1 + kt_2} \quad (4.3)$$

where

$$a_0 = f_1 + f_2 k + f_3 k^2$$

F
2
5

TABLE 1.- VALUES OF ω_{min} FOR A CLOSED CONICAL SHELL

δ / b	0.03	0.02	0.015	0.01	0.009	0.008	0.007
$\alpha = 3°$	—	—	—	—	—	0.102	0.0931
5	—	—	—	0.141	0.133	0.126	0.116
10	—	0.281	0.242	0.193	0.184	0.175	0.162
15	0.419	0.335	0.288	0.236	0.223	0.212	0.201
20	0.479	0.381	0.355	0.287	0.267	0.249	0.231
25	0.519	0.432	0.369	0.311	0.293	0.275	0.258
30	0.562	0.467	0.404	0.337	0.319	0.302	0.286
35	0.607	0.499	0.438	0.362	0.344	0.328	0.312
40	0.652	0.529	0.469	0.386	0.368	0.351	0.335
45	0.693	0.559	0.498	0.408	0.390	0.372	0.355
50	0.729	0.586	0.522	0.430	0.409	0.390	0.372
55	0.757	0.614	0.541	0.452	0.429	0.406	0.386
60	0.776	0.644	0.582	0.479	0.451	0.424	0.399
65	0.789	0.688	0.582	0.493	0.478	0.450	0.418
70	0.809	0.695	0.627	0.504	0.482	0.462	0.444
75	0.877	0.701	0.628	0.548	0.515	0.483	0.453
80	0.891	0.810	0.672	0.553	0.533	0.514	0.434
85	0.963	0.779	0.703	0.643	0.634	0.625	0.589
87	—	0.988	0.818	0.671	0.645	0.622	0.600

δ / b	0.006	0.005	0.004	0.003	0.002	0.001	0.0008
3	0.0897	0.0769	0.0677	0.0580	0.0474	0.0340	0.0311
5	0.106	0.0967	0.0877	0.0748	0.0623	0.0448	0.0408
10	0.149	0.138	0.124	0.108	0.0895	0.0656	0.0590
15	0.186	0.169	0.154	0.134	0.112	0.0814	0.0736
20	0.214	0.199	0.178	0.157	0.130	0.0950	0.0862
25	0.244	0.223	0.200	0.177	0.148	0.107	0.0968
30	0.271	0.244	0.222	0.193	0.161	0.119	0.107
35	0.288	0.264	0.242	0.209	0.174	0.128	0.116
40	0.307	0.282	0.259	0.224	0.187	0.138	0.125
45	0.326	0.299	0.274	0.238	0.198	0.147	0.133
50	0.345	0.315	0.288	0.251	0.210	0.156	0.141
55	0.366	0.331	0.299	0.264	0.223	0.165	0.149
60	0.376	0.350	0.312	0.279	0.231	0.172	0.157
65	0.388	0.361	0.331	0.288	0.243	0.182	0.166
70	0.414	0.376	0.341	0.307	0.256	0.194	0.176
75	0.425	0.400	0.367	0.319	0.276	0.207	0.190
80	0.480	0.432	0.387	0.349	0.298	0.229	0.211
85	0.537	0.488	0.445	0.408	0.358	0.283	0.256
87	0.581	0.564	0.550	0.472	0.400	0.322	0.297

TABLE 2.- VALUES OF THE NUMBER OF HALF-WAVES $\underline{n}$ OF OSCILLATION OF A CLOSED CONICAL SHELL

s/b	0.03	0.02	0.015	0.01	0.009	0.008	0.007
$\alpha = 3$	—	—	—	—	—	2	2
5	—	—	—	2	2	2	3
10	—	2	3	3	3	4	4
15	2	3	3	4	4	4	4
20	3	3	4	5	5	5	5
25	3	3	4	5	5	5	5
30	3	4	4	5	5	5	5
35	3	4	4	5	5	5	5
40	3	4	4	5	5	5	6
45	3	4	4	5	5	5	5
50	3	4	4	5	5	5	5
55	3	4	4	5	5	5	5
60	3	4	4	5	5	5	5
65	3	4	4	4	4	5	5
70	3	3	4	4	4	4	4
75	3	3	3	4	4	4	4
80	2	3	3	3	3	3	3
85	2	2	2	2	2	2	3
87	—	2	2	2	2	2	2

s/b	0.006	0.005	0.004	0.003	0.002	0.001	0.0008
3	3	2	3	3	4	5	5
5	3	3	4	4	5	6	6
10	4	4	5	5	6	7	8
15	5	5	5	6	7	8	9
20	5	5	6	6	7	9	9
25	5	6	6	7	8	9	10
30	6	6	6	7	8	9	10
35	6	6	6	7	8	10	10
40	6	6	7	7	8	10	10
45	6	6	6	7	8	10	10
50	6	6	6	7	8	9	10
55	6	6	6	7	8	9	10
60	5	6	6	6	7	9	9
65	5	5	6	6	7	8	9
70	5	5	5	6	6	8	8
75	4	4	5	5	6	7	7
80	4	4	4	4	5	6	6
85	3	3	3	3	4	5	5
87	2	2	2	3	3	4	4

$$-a_1 = \underline{(\pi^2m^2\lambda^2+n^2)^2} + \underline{(3+2\mu)\pi^2m^2\lambda^2} + \underline{n^2} + k\Big[\underline{\frac{3-\mu}{1-\mu}(\pi^2m^2\lambda^2+n^2)^3} + 2\pi^2m^2\lambda^2\Big(2\pi^2m^2\lambda^2+2-\frac{2-\mu^2}{1-\mu}n^2\Big) - \frac{3+\mu}{1-\mu}n^4 + \frac{2}{1-\mu}n^2\Big] + 2k^2\pi^4m^4\lambda^4[2\pi^2m^2\lambda^2+(1+\mu)n^2]$$

F
2
5

$$a_2 = \underline{\frac{3-\mu}{1-\mu}(\pi^2m^2\lambda^2+n^2)} + \underline{\frac{2}{1-\mu}} + \underline{2k}\Big[\underline{\frac{1}{1-\mu}(\pi^2m^2\lambda^2+n^2)^2} + 2\pi^2m^2\lambda^2 + \frac{n^2}{1-\mu}\Big]$$

$$a_3 = -\underline{\frac{2}{1-\mu}}$$

$$f_1 = (1-\mu^2)\pi^4m^4\lambda^4$$

$$f_2 = (\pi^2m^2\lambda^2+n^2)^4 - (2n^2-1)(2\pi^2m^2\lambda^2+n^2)^2 + 2\pi^4m^4\lambda^4\mu^2(n^2-2)$$

$$f_3 = \pi^4m^4\lambda^4[(2\pi^2m^2\lambda^2+n^2)^2-\mu^2n^4]$$

$$t_1 = (\pi^2m^2\lambda^2+n^2), \qquad t_2 = (2\pi^2m^2\lambda^2+n^2)^2 + \frac{2\mu^2}{1-\mu}\pi^2m^2\lambda^2n^2 \qquad (4.4)$$

In the formulas (4.4) the underlined members represent the solution of W. Flügge [8,9].

Retaining only the principal members in (4.3) we find

$$\omega^{*2} = \frac{(1-\mu^2)\pi^4m^4\lambda^4}{(\pi^2m^2\lambda^2+n^2)^2} + k(\pi^2m^2\lambda^2+n^2)^2 \qquad (4.5)$$

which agrees with the formula obtained in the literature [10] by another method.

Formulas (4.2), (4.3) and (4.5) determine the frequency of natural oscillations of a closed elastic circular cylindrical shell fastened at both ends.

The oscillations of cylindrical shells in particular have been investigated [11-14], and in one paper [13] an analysis was made of a series of results of solutions in which there are mentioned works not mentioned in our bibliographical list.

A. P. Philippov [11] in a numerical example previously studied by Flügge [8,9] pointed out that for fairly large values of λ the frequencies of the natural oscillations of a cylindrical shell with rigid fastenings and rigidly attached rims are practically the same.

5. <u>Comments</u>. First of all we observed two cases investigated by V. E. Breslavskii [4]. In the first a = 10 cm, b = 17.5 cm, l = 60 cm, δ = 0.1 cm; as a result the mean radius of the conical shell equals $R = 1/2\,(a + b) = 13.75$ cm, the angle of inclination of the generator to the base $\alpha = 82^\circ\,13'$, the height of the conical shell will be $h = l \sin\alpha = 59.44$ cm.

In the second case a = 12.5 cm, $\underline{b}$ = 15 cm, $\underline{l}$ = 85 cm, δ = 0.1 cm and R = = 13.75 cm, $\underline{h}$ = 84.97 cm, $\alpha = 88^\circ\ 19'$. The mechanical characteristics of the material were not given in the literature [4].

Let us assume that they were the same as in another paper of that author [14]: $E = 1.93 \times 10^6$ kg/cm^2; $\mu = 0.3$; $\gamma = 0.00785$ kg/cm^3. We will take g = 981 cm/ sec^2.

It is evident that the two conical shells differed little from the cylindrical. Let us therefore interpolate in place of a conical shell a cylindrical shell equivalent to it, with a radius of the middle surface equal to the half-sum of the radii of the bases of the conical shell, with a length equal to the height of the conical shell, and with the same thickness. Then for the two equivalent cylindrical shells $\underline{k} = 4.4077 \times 10^{-6}$, and the ratio of the radius to the length will be = 0.2313 and = 0.1618 respectively. We calculate according to formula (4.5). The lowest frequency will be at $\underline{m}$ = 1. If $\underline{f} = \omega/2\pi$ for the first case the lowest frequency will be at $\underline{n}$ = 4, whereupon $\underline{f}_1$ = 254 osc/sec, $\underline{f}_2$ = 275 osc/sec, $\underline{f}_3$ = 285 osc/sec; here $\underline{f}_1$ and $\underline{f}_3$ = the calculated and experimental values of the frequency obtained from the literature [4], $\underline{f}_2$ = the value of the frequency calculated according to formula (4.5). For the second example ($\underline{n}$ = 3) $\underline{f}_1$ = 192 osc/sec, $\underline{f}_2$ = 196 osc/sec, $\underline{f}_3$ = 200 osc/sec.

The number of half-waves on the perimeter of the shell agrees in the two cases.

Thus the unnecessary complication introduced into the calculation of a conical shell similar to the cylindrical was unwarranted. The calculations show that when $\gtrapprox$ 75 the above-suggested substitution for the conical shell of the equivalent cylindrical is always allowable with quite insignificant error within the limits of accuracy of the determination of the frequency of natural oscillations.

Let us analyze the numerical examples given by Urk and Hut [3]. The dimensions of the shells examined by them are given in Table 3. The Poisson's ratio of the material of the shell equals $\mu = 0.3$, $\gamma/gG = 39.4 \times 10^{-11}$ sec^2.

TABLE 3

Case	a, cm	h, cm	$\delta\ 10^3$ cm	λ	$\psi\ 10^6$	α°	f_1 osc/sec	f_2 osc/sec
1	2.45	4.05	20	0.605	203	32.5	140	250
2	2.45	4.05	11.4	0.605	66.0	32.5	80	150
3	2.45	4.05	7.8	0.605	30.9	32.5	54.7	120
4	2.45	4.05	6.4	0.605	20.8	32.5	45.0	—
5	2.45	4.05	4.2	0.605	8.96	32.5	29.5	—
6	0	5.61	11.4	0	34.5	32.5	—	—
7	2.45	4.05	11.4	0.605	66.2	32.5	80	150
8	3.9	3.12	11.4	1.25	111	32.5	51	—
9	5.3	2.23	11.4	2.38	218	32.5	48	—

According to Strutt [1]:

$$\omega_n^2 = \frac{1}{6}\frac{Eg}{(1+\mu)\gamma}\frac{\delta^2}{a^4}\left\{\frac{1}{2(1-\mu)}\frac{(n^3-n)^2}{\cos^2\alpha}\left[\ln\frac{1}{q}-\frac{1}{2}(1-q)(3-q)\right]-\right.$$
$$\left.-\frac{1}{2}\sin^2\alpha(1-q^2)(4n^2-\sin^2\alpha)+n^4\ln\frac{1}{q}\right\}:$$
$$:\left\{(n^2+1)\left[\frac{1}{4}\left(\frac{1}{q^4}-1\right)+\frac{1}{2}\left(\frac{1}{q^2}-1\right)-\frac{2}{3}\left(\frac{1}{q^3}-1\right)\right]\mathrm{tg}^2\alpha+\right.$$
$$\left.+\frac{1}{4}n^2\left(\frac{1}{q^4}-1\right)-\frac{2}{3}\left(\frac{1}{q^3}-1\right)(n^2-1)+\frac{1}{2}\left(\frac{1}{q^2}-1\right)\left(\frac{1}{n^2}+n^2-2\right)\right\} \qquad \left(q=\frac{a}{b}\right) \quad (5.1)$$

The calculations for the indicated examples using formula (5.1) show that the lowest values of the frequencies of natural oscillations of the shell always occur at $n = 2$. They are given in Table 3, where f_1 and f_2 correspond to the values found by formula (5.1) and established experimentally.

As an example let us determine the lowest frequencies of natural oscillations of a truncated conical shell in which $\delta/a = 0.001$, $\lambda = a/h = 0.5$, $\alpha = 40, 50, 60, 70°$ ($= 0.3$).

We calculate using formula (2.7). We obtain accordingly $10\omega^2_{*min} =$ $= 837$ ($n = 16$), 957 ($n = 15$), 866 ($n = 14$), 620 ($n = 12$).

The non-conformity of the experimental data with the calculations given in Table 3 is explained also by a difference existing in the limiting conditions of the problem. The oscillation of a conical shell, firmly closed at one end and free at the other, can be the subject of special study.

REFERENCES

1. Strutt M.O.I., Eigenschwingungen einer Kegelschale, Annalen der Physik, Series 5, Vol. 17, Nr. 7, pp. 729-735, 1933 (In German: Natural Oscillation of a Conical Shell).

2. Rayleigh, J.W. Teoriya zvuka, vol. 1 Gostekhteoretizdat, 1940 (In Russian: The Theory of Sound).

3. Urk A. T. van, and Hut G. B., Messung derRadialschwingungen von Aluminium Kegelschale, Annalen der Physik, Series 5, Vol. 17, Nr. 8, pp. 915-920, 1933 (In German: The Measurement of Radial Oscillation of Aluminum Conical Shells).

4. Breslavskii V. E., Nekotorye osnovnye sluchai svobodnykh kolebanii konicheskikh obolochek. Trudy Khar'kovskovo Vysshevo aviatsionno-inzhinernovo voennovo uchilishcha, Nr. 27, pp. 107-123, 1954. (In Russian: Certain basic instances of free oscillations of conical shells).

5. Niordson F.I.N., Buckling of conical shells subjected to uniform external lateral pressure. Transactions of the Royal Institute of Technology, Stockholm, Nr. 10, p. 21, 1947.

6. Grigolyuk E. I., Ob ystoichivosti zamknutoi dvusloinoi konicheskoi obolochki pod deistviem ravnomernovo normal'novo davleniya. Inzhinernii sbornik, vol. 19, pp. 73-82, 1954 (In Russian: On the stability of closed double-layer conical shells under the influence of a steady normal pressure).

7. Grigolyuk E. I., Uprugaia ustoichivost' ortotropnykh u sloistykh konicheskikh u tsilindricheskikh obolochek. Sb. Prostranstvennye sistemy, vol. 3, pp. 375-421, Gosstroiizdat, 1955 (In Russian: Elastic stability of orthotropic and laminated conical and cylindrical shells).

8. Flügge W., Schwingungen zylindrischer Schalen, Zeitschrift für angewandte Mathematik und Mechanik, vol. 17, Nr. 6, pp. 425-527, 1933. (In German: Oscillation of cylindrical shells).

9. Flügge W., Statik un Dynamik der Schalen, Berlin, J. Springer Verlag, pp. 227-232, 1934 (In German: The Statics and Dynamics of Shells).

10. Grigolyuk E. I., K raschetu na prochnost' tsilindricheskoi bimetallicheskoi obolochki pri nagruzke, zavisiashchei ot vremeni. Tr. Gos. soyuzn. n.-i. instituta (Red. V. S. Pankratov), Nr. 2 (6), pp. 91-106, 1954 (In Russian: Calculation of the stability of cylindrical bimetallic shells under pressure, based upon time).

11. Philippov A. P., Kolebaniya tsilindricheskikh obolochek. PMM, Novaya Seriya, vol. 1, Nr. 2, pp. 177-186, 1937. (In Russian: Oscillations of cylindrical shells).

12. Vlasov, V, Z., Osnovnye uravneniya obshchei teorii uprugikh obolochek, PMM, vol. 8, Nr. 2, pp. 109-140, 1944 (In Russian: Basic equations of the theory of elastic shells).

13. Naghdi P. M., Berry J. G., On the equations of motion of cylindrical shells. Journal of applied mechanics, vol. 21, No. 2, pp. 160-166, 1954.

14. Breslavskii V. E., O kolebaniiakh tsilindricheskikh obolochek. Inzhinernyi sb., vol. 16, pp. 109-118, 1953 (In Russian: The oscillations of cylindrical shells).

Translated by Consultants Custom Translations, Inc.,
227 West 17th Street,
New York 11, New York.